PHYX412-1 Fall 2008 : Quantum Mechanics I

Homework Assignment 6 : Angular Momentum

1. Spin 3/2

A particle has internal spin of 3/2 and carries no orbital angular momentum.

A. What values of $\hat{J}_z$ are allowed by measurements? What is the value of $\hat{J}^2$?

B. Construct the matrix elements of $\hat{J}_x$, $\hat{J}_y$, and $\hat{J}_z$ in the basis of states with definite j and m.

2. Rotations of a Spin 1 Particle

A particle has internal spin of 1 and carries no orbital angular momentum. As we saw in class, a general rotation can be written as a product of rotations about the y- and z-axes by the Euler angles,

$$\hat{D}(\alpha, \beta, \gamma) \;=\; \hat{D}_z(\alpha)\hat{D}_y(\beta)\hat{D}_z(\gamma)$$

A. Derive the matrix elements $\langle 1, m' | \hat{D}(\alpha, \beta, \gamma) | 1, m \rangle$.

B. We measure the particle's $\hat{J}_z$ and find $m = 0$. Immediately afterward, we measure the component of spin along the axis defined by the Euler angles α, β, γ (that is to say, along the axis we obtain when we apply the rotation by α, β, γ to the z-axis). What are the possible outcomes of this measurement and the probabilities to obtain each one?

3. Orbital Angular Momentum

A spin zero particle is in state $|\psi\rangle$ which has an angular wave function given by,

$$\langle \vec{x} | \psi \rangle \;=\; \mathcal{N} \cos^2 \theta$$

Determine the normalization $\mathcal{N}$. What are the possible outcomes of a measurement of $\hat{L}_z$ and their probabilities? How about for $\hat{L}^2$?

Hint: If you need to look them up, you can find a table of normalized spherical harmonics online:
`http://en.wikipedia.org/wiki/Table_of_spherical_harmonics`
Staring at it before you begin can save calculating a lot of integrals!

4. Angular Momentum and Angular Position

Define an operator $\hat{\phi}$ which measures a particles ϕ coordinate:

$$\hat{\phi}|\theta', \phi'\rangle = \phi'|\theta', \phi'\rangle$$

where $|\theta', \phi'\rangle$ are position eigenkets with definite θ and ϕ. If $\hat{R}$ is an infinitesimal rotation about the z axis by amount $\delta \ll 1$, compute $\hat{\phi}\hat{R}|\theta', \phi'\rangle$ and $\hat{R}\hat{\phi}|\theta', \phi'\rangle$ (you need work only to first order in δ). Take the difference of the two equations, drop all terms of order δ^2, and thus derive the commutator $[\hat{\phi}, \hat{L}_z]$.